MULTIGRID METHOD FOR WILSON NONCONFORMING FINITE ELEMENT WITH NUMERICAL INTEGRATION(Domain Decomposition Methods and Related Topics)

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MULTIGRID METHOD FOR WILSON NONCONFORMING FINITE ELEMENT WITH NUMERICAL INTEGRATION*)

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Summary. In this paper, an effective multigrid algorithm is applied to the Wilson nonconforming finite element, which has been extensively used to solve the second-order elliptic boundary value problems. We obtain good convergence rates for the V-cycle multigrid method with or without numerical quadratures.

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1.Introduction

The Wilson nonconforming finite element has been widely used in computational mechanics and structural engineering because of its good convergence behavior. It is shown in [10],[12], that the convergence rate of Wilson element in the energy norm is of first order. The condition number of its stiff matrix is $O(h^{-2})$, resulting in a slow convergence rate in actual computations. Therefore PCG or other preconditioned iterative methods must be carried out to speed up the convergence.

As we know, the multigrid method is a useful tool to solve linear systems arising from the discretization of elliptic boundary value problems and can produce some good preconditioners. We refer to [1,2,3,4,6,9,11] and references therein for a comprehensive treatment of this method. However, most of multigrid methods is based on the conforming finite element approximation. In the case of nonconforming elements, we must construct an intergrid transfer operator between fine and coarse grids. On the other hand, the stiffness matrix of a conforming or nonconforming finite element discritization is usually computed approximately using a suitable quadrature scheme. The effect of numerical integration in finite element methods was analyzed in [7], where only conforming elements are concerned. Based on the idea of [7], the V-cycle multigrid algorithm with numerical integration on each grid level was analyzed in [8]. It was proved there that the constructed preconditioner has a uniform convergence rate for the approximation of problems with a full regularity and a quasi-uniform mesh, just the same as without using numerical integration.

However, there are no relevent results of the effect of numerical integration for nonconforming finite elements. In this paper, an effective multigrid algorithm is applied to the Wilson nonconforming element. We obtain good convergence rates for the V-cycle multigrid method with or without numerical integration.

We organize the paper as follows. In section 2, the error estimate of Wilson element approximation using numerical integration is obtained. In section 3, we consider a multigrid algorithm for the Wilson element. Two intergrid tranfer operators are constructed, which produce good preconditioners. Section 2 and Section 3 are independent. Based on the results of Section 2, Section 3 and

This paper is dedicated to Professor Hideo Kawarada on the occasion of his 60th birthday.

[13], we apply the multigrid algorithm to the Wilson element in Section 4, when the quadrature schemes of Section 2 are used. We obtain the optimal preconditioners as those in Section 3 without using numerical integration.

2. Effect Of Numerical Integration On Wilson Element.

It's shown in [7] that when a suitable quadrature scheme is used for the bilinear element approximation, the first-order convergence rate can be guaranteed. In this section, we prove that this first-order convergence rate can also be obtained when the same quadrature scheme is used for Wilson element.

We consider the general second-order elliptic boundary value problem

$$\begin{cases} -\{\partial_x(a_{11}\partial_x u) & +\partial_y(a_{12}\partial_x u)+\partial_x(a_{12}\partial_y u) \\ & +\partial_y(a_{22}\partial_y u)\}+au=f \quad \text{in} \quad \Omega, \\ u & =0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where all funtions $a_{ij}, i,j=1,2,a$ and f are smooth enough, and we assume that the differential operator is uniformly elliptic, i.e., there is a positive constant c such that

$$\begin{aligned} c^{-1}(\xi_1^2+\xi_2^2) &\le a_{11}\xi_1^2+2a_{12}\xi_1\xi_2+a_{22}\xi_2^2 \\ &\le c(\xi_1^2+\xi_2^2), \\ a &\ge 0 \end{aligned}$$

for all $x\in\overline{\Omega}$ and real ξ_1,ξ_2.

Let $\mathcal{J}_h$ be a rectangular partition of Ω, satisfying the regularity assumption [2], $z_0=(x_0,y_0)$ is the center of $K\in\mathcal{J}_h$, $2h_x$ and $2h_y$ are the lengths of two edges of K in x and y direction respectively, $h=max_K(h_x,h_y)$.

The variational problem of (2.1) is to find $u\in H_0^1(\Omega)$ such that

$$\overline{A}(u,v)=(f,v) \quad \text{for all} \quad v\in H_0^1(\Omega), \tag{2.2}$$

where

$$\begin{aligned} \overline{A}(u,v)=\int_\Omega &[a_{11}\partial_x u\partial_x v+a_{12}(\partial_x u\partial_y v+\partial_y u\partial_x v) \\ &+a_{22}\partial_y u\partial_y v+auv]dxdy \quad \text{for all} \quad u,v\in H_0^1(\Omega). \end{aligned}$$

The Wilson element solution $w^*\in W_h$ of (2.2) satisfies

$$\overline{A}_h(w^*,v)=(f,v) \quad \text{for all} \quad v\in W_h, \tag{2.3}$$

where

$$\overline{A}_h(u,v)=\sum_{K\in\mathcal{J}_h}\iint_K(a_{11}\partial_x u\partial_x v+a_{12}\partial_x u\partial_y v++a_{21}\partial_y u\partial_x v+a_{22}\partial_y u\partial_y v+auv)dxdy,$$

and the finite element space $W_h=\{w_h, w_h|_K\in P_2(K)$ is determined by the function values at the four vertices of K and the mean values of its two second derivatives $\partial_{xx}w_h$ and $\partial_{yy}w_h$ on K, $w_h=0$ at vertices belonging to $\partial\Omega\}$.

The bilinear element solution $u^*\in BL_h$ satisfies

$$\overline{A}_h(u^*,v)=(f,v) \quad \text{for all} \quad v\in BL_h, \tag{2.4}$$

where $BL_h = \{u_h, u_h|_K \in Q_1(K)$ is determined by its function values at four vertices of K, $u_h|_{\partial\Omega} = 0\}$.

In the following, we assume that c(with or without a subscript) is a generic constant which may take different values at different places and is independent of the mesh size h and the solution u.

The following lemmas are known or can be easily derived.

Lemma 2.1.[7].

$$|u - u^*|_{1,h} \le ch\|u\|_{2,\Omega}, \tag{2.5}$$

$$|u - u^*|_{0,\Omega} \le ch^2\|u\|_{2,\Omega}, \tag{2.6}$$

where the semi-norm

$$|\cdot|_{1,h} = (\sum_K |\cdot|_{1,K}^2)^{\frac{1}{2}}.$$

Lemma 2.2.[10].

$$|u - w^*|_{1,h} \le ch\|u\|_{2,\Omega}, \tag{2.7}$$

$$|u - w^*|_{0,\Omega} \le ch^2\|u\|_{2,\Omega}. \tag{2.8}$$

We approximate the exact integrals in $\overline{A}_h(u,v)$ by defining a quadrature scheme Q_K over each element $K \in \mathcal{J}_h$. To be specific, we first consider the reference rectangle $\hat{K}$ and approximate the integral $\int_{\hat{K}} \hat{\phi}(\hat{x})d\hat{x}$ as follows:

$$\int_{\hat{K}} \hat{\phi}(\hat{x})d\hat{x} \approx \sum_{l=1}^{L} w_l \hat{\phi}(b_l),$$

where w_l are positive weights and $b_l \in \hat{K}$ are quardrature points. We then define the quadrature rule on each K by

$$\int_K \phi(x) \approx \sum_{l=1}^{L} w_{K,l}\phi(b_{K,l}) \equiv Q_K[\phi],$$

where $\phi(x) = \hat{\phi}(\hat{x})$, the weights $w_{K,l}$ and quadrature points $b_{K,l}$ are defined in terms of the w_l and b_l by means of the affine mapping B_K from K onto $\hat{K}$ that takes each x in K into $\hat{x}$ in $\hat{K}$..

The quadrature error functional is denoted by

$$E_K[\phi] \equiv \int_K \phi(x)dx - Q_K[\phi] = det(B_k)\hat{E}[\hat{\phi}]. \tag{2.9}$$

Using the quadrature scheme, we approximate $\overline{A}_h(.,.)$,$(f,.)$ by $A_h(.,.)$,$(f,.)_h$ as follows:

$$\begin{aligned} A_h(u,v) \equiv \sum_{K\in\mathcal{J}_h} Q_K[&a_{11}\partial_x u\partial_x v + a_{12}(\partial_x u\partial_y v + \partial_y u\partial_x v) \\ &+ a_{22}\partial_y u\partial_y v + auv], \end{aligned} \tag{2.10}$$

$$(f,v)_h \equiv \sum_{K\in\mathcal{J}_h} Q_K(fv).$$

Now we define the Wilson and the bilinear element solution w_h and u_h with the quadrature scheme Q_K being used in the approximation of (2.3) and (2.4):

$$A_h(w_h, v) = (f,v)_h \quad \text{for all} \quad v \in W_h, \tag{2.11}$$

$$A_h(u_h, v) = (f,v)_h \quad \text{for all} \quad v \in BL_h. \tag{2.12}$$

Following [7], we use three assumptions in the deriviton of quadrature schemes.
Assumption 1. The union of all quadrature points b_l on $\hat{K}$ contains a $P_1(\hat{K})$ unisolvent subset.
Assumption 2. The quadrature scheme Q_K satisfies:

$$E_K[\phi] \equiv 0 \quad \text{for all} \quad v \in Q_1(K).$$

Assumption 3. The quadrature scheme Q_K satisfies:

$$\text{the weights } w_{K,l} > 0.$$

It will be seen later that by a proper choice of b_l and w_l, there exist schemes satisfying all three assumptions.

The following lemma states a convergence result for the bilinear element solution of (2.12).

Lemma 2.3[7]. Suppose $a_{ij}, a \in W^{1,\infty}(\Omega), f \in W^{1,q}(\Omega), q > 2$, and the quadrature scheme satisfies Assumption 1,2,3. Then

$$|u - u_h|_{1,h} \leq ch[(\sum_{i,j=1}^{2} ||a_{ij}||_{1,\infty} + ||a||_{1,\infty})||u||_2 + |u|_2 + |f|_{1,q}],$$

where u, u_h are the solution of (2.2),(2.12) respectively.

Now we prove that the similar result holds for the Wilson element.

Theorem 2.1. Suppose $a_{ij}, a \in W^{1,\infty}(\Omega), f \in W^{1,q}(\Omega), q > 2$, and the quadrature scheme satisfies Assumption 1,2,3. Then

$$|u - w_h|_{1,h} \leq ch[(\sum_{i,j=1}^{2} ||a_{ij}||_{1,\infty} + ||a||_{1,\infty})||u||_2 + |u|_2 + |f|_{1,q}],$$

where u, w_h are the solution of (2.2),(2.11) respectively.

The proof will be given later. Before proving Theorem 2.1, we need some lemmas.

Lemma 2.4.

$$|u - w_h|_{1,h} \leq c[inf_{v_h \in W_h}(||u - v_h||_{1,h} + sup_{w_h \in W_h} \frac{|\overline{A}_h(v_h, w_h) - A_h(v_h, w_h)|}{||w_h||_{1,h}}) + sup_{w_h \in W_h} \frac{|f(w_h) - f_h(w_h)|}{||w_h||_{1,h}} + sup_{w_h \in W_h} \frac{|\overline{A}_h(u, w_h) - f(w_h)|}{||w_h||_{1,h}}]$$

Lemma 2.4 can be proved by using nearly the same arguments as in the proof of the first Strang Lemma [7].

Lemma 2.5[7].

$$inf_{v_h \in W_h} ||u - v_h||_{1,h} \leq ||u - \Pi_h u||_{1,h} \leq ch||u||_2,$$

where $\Pi_h u$ is the interpolant of u in W_h.

Lemma 2.6[7].

$$sup_{w_h \in W_h} \frac{|\overline{A}_h(u, w_h) - f(w_h)|}{||w_h||_{1,h}} \leq ch||u||_2.$$

Lemma 2.5 and 2.6 are, respectively, the interpolation and the consistency error estimate of Wilson nonconforming element.

Lemma 2.7. Suppose the quadrature scheme satisfies Assumption 1,2,3, then there exists a constant c independent of $K \in \mathcal{J}_h$ and h such that for all $a \in W^{1,\infty}(K), p, p' \in P_2(K), i, j = x, y$

$$|E_K(a\partial_i p \partial_k p')| \leq ch_k \|a\|_{1,\infty,K} \|p\|_{1,K} \|p'\|_{2,K}, \tag{2.13}$$

$$|E_K(app')| \leq ch_k \|a\|_{1,\infty,K} \|p\|_{1,K} \|p'\|_{1,K}. \tag{2.14}$$

Proof. (1) Let $v = \partial_k p', w = \partial_i p, \phi = av$, then $v, w \in Q_1(K)$. We have

$$E_K(avw) = det(B_K)\hat{E}(\hat{a}\hat{v}\hat{w}),$$

$$\begin{aligned}
|\hat{E}(\hat{a}\hat{v}\hat{w})| &= \hat{E}(\hat{\phi}\hat{w}) \\
&= |\int_{\hat{K}} \hat{\phi}\hat{w} d\hat{x} - \sum_{l=1}^{L} w_l(\hat{\phi}\hat{w})(b_l)| \\
&\leq \hat{c}|\hat{\phi}\hat{w}|_{0,\infty,\hat{K}} \\
&\leq \hat{c}\|\hat{\phi}\|_{1,\infty,\hat{K}} |\hat{w}|_{0,\hat{K}}.
\end{aligned}$$

According to Assumption 2,

$$\hat{E}(\hat{\phi}\hat{w}) = 0 \quad \text{for all} \quad \hat{\phi} \in P_0(K).$$

Then applying Bramble-Hilbert Lemma gives

$$|\hat{E}(\hat{\phi}\hat{w})| \leq \hat{c}|\hat{\phi}|_{1,\infty,\hat{K}} |\hat{w}|_{0,\hat{K}},$$

that is,

$$|\hat{E}(\hat{a}\hat{v}\hat{w}) \leq \hat{c}(|\hat{a}|_{0,\infty,\hat{K}} |\hat{v}|_{1,\hat{K}} + |\hat{a}|_{1,\infty,\hat{K}} |\hat{v}|_{0,\hat{K}}) |\hat{w}|_{0,\hat{K}}.$$

Therefore,

$$\begin{aligned}
|E_K(avw)| &\leq \hat{c}det(B_K)(\sum_{i=0}^{1} |\hat{a}|_{i,\infty,\hat{K}} |\hat{v}|_{1-i,\hat{K}}) |\hat{w}|_{0,\hat{K}} \\
&\leq ch_k \|a\|_{1,\infty,K} \|v\|_{1,K} |w|_{0,K}.
\end{aligned}$$

(2.13) follows by replacing v, w with $\partial_k p', \partial_i p$ in the last inequality.
(2) Let $\phi = ap'$, we have

$$E_K(ap'p) = det(B_K)\hat{E}(\hat{a}\hat{p}'\hat{p}). \tag{2.15}$$

Let $\hat{\Pi}$ denote the orthogonal projection from the space $L^2(\hat{K})$ onto the subspace $Q_1(\hat{K})$. Then

$$\begin{aligned}
\hat{E}(\hat{a}\hat{p}'\hat{p}) &= \hat{E}(\hat{\phi}\hat{p}) \\
&= \hat{E}(\hat{\phi}\hat{\Pi}\hat{p}) + \hat{E}(\hat{\phi}(\hat{p} - \hat{\Pi}\hat{p})).
\end{aligned} \tag{2.16}$$

Using the same technique as in the first part (1) and the fact that $|\hat{\Pi}\hat{p}|_{0,\hat{K}} \leq |\hat{p}|_{0,\hat{K}}$, we get

$$\hat{E}(\hat{\phi}\hat{\Pi}\hat{p})| \leq c(\sum_{i=0}^{1} |\hat{a}|_{i,\infty,\hat{K}} |\hat{p}'|_{1-i,\hat{K}}) |\hat{p}|_{0,\hat{K}} \tag{2.17}$$

From the definition of $\hat{\Pi}$, we have

$$|\hat{p} - \hat{\Pi}\hat{p}|_{0,\hat{K}} \leq \hat{c}|\hat{p}|_{1,\hat{K}}.$$

Therefore,

$$\begin{aligned}|\hat{E}(\hat{\phi}(\hat{p} - \hat{\Pi}\hat{p}))| &\leq \hat{c}|\hat{\phi}(\hat{p} - \hat{\Pi}\hat{p})|_{0,\infty,\hat{K}} \\ &\leq \hat{c}|\hat{\phi}|_{0,\infty,\hat{K}}|\hat{p} - \hat{\Pi}\hat{p}|_{0,\hat{K}} \\ &\leq \hat{c}|\hat{a}|_{0,\infty,\hat{K}}|\hat{p}'|_{0,\hat{K}}|\hat{p}|_{1,\hat{K}}.\end{aligned} \tag{2.18}$$

Combining (2.16),(2.17),(2.18) with (2.15) yields

$$\begin{aligned}|E_K(app')| &\leq cdet(B_K)[(\sum_{i=0}^{1}|\hat{a}|_{i,\infty,\hat{K}}|\hat{p}'|_{1-i,\hat{K}})|\hat{p}|_{0,\hat{K}} \\ &+ |\hat{a}|_{0,\infty,\hat{K}}|\hat{p}'|_{0,\hat{K}}|\hat{p}|_{1,\hat{K}}] \\ &\leq ch_k||a||_{1,\infty,K}||p||_{1,K}||p'||_{1,K}.\end{aligned}$$

Lemma 2.7 immediately implies Lemma 2.8.

Lemma 2.8. Suppose $a_{ij}, a \in W_{1,\infty}(\Omega), u \in H^2(\Omega)$, and the quadrature scheme satisfies Assumption 1,2,3.Then

$$|\overline{A}_h(\Pi_h u, w_h) - A_h(\Pi_h u, w_h)| \leq ch(\sum_{i,j=1}^{2}||a_{ij}||_{1,\infty} + ||a||_{1,\infty})||w_h||_{1,h}||u||_2.$$

Lemma 2.9. Suppose the quadrature scheme satisfies Assumption 1,2,3. Then there exists a constant c independent of $K \in \mathcal{J}_h$ and h such that for all $f \in W_{1,q}(K), p \in P_2(K)$,

$$|E_K(fp)| \leq ch_k|det(B_K)|^{\frac{1}{2}-\frac{1}{q}}||f||_{1,q,K}||p||_{1,K}.$$

Proof.

$$E_K(fp) = det(B_K)\hat{E}(\hat{f}\hat{p}).$$

Like (2.16), we have

$$\hat{E}(\hat{f}\hat{p}) = \hat{E}(\hat{f}\hat{\Pi}\hat{p}) + \hat{E}(\hat{f}(\hat{p} - \hat{\Pi}\hat{p})), \tag{2.20}$$

where

$$\begin{aligned}|\hat{E}(\hat{f}\hat{\Pi}\hat{p})| &\leq \hat{c}|\hat{f}\hat{\Pi}\hat{p}|_{0,\infty,\hat{K}} \\ &\leq c||\hat{f}||_{1,q,\hat{k}}|\hat{p}|_{0,\hat{K}}.\end{aligned}$$

¿From assumption 2, we have

$$\hat{E}(\hat{f}\hat{\Pi}\hat{p})| = 0 \quad \text{for all} \quad \hat{f} \in P_0(\hat{K}).$$

Thus Bramble-Hilbert Lemma gives

$$|\hat{E}(\hat{f}\hat{\Pi}\hat{p})| \leq c|\hat{f}|_{1,q,\hat{K}}|\hat{p}|_{0,\hat{K}}. \tag{2.21}$$

On the other hand,

$$
\begin{aligned}
|\hat{E}(\hat{f}(\hat{p}-\hat{\Pi}\hat{p}))| &\le c|\hat{f}|_{0,q,\hat{K}}|\hat{p}-\hat{\Pi}\hat{p}|_{0,\hat{K}} \\
&\le c|\hat{f}|_{0,q,\hat{K}}|\hat{p}|_{1,\hat{K}}.
\end{aligned} \tag{2.22}
$$

Therefore, by applying (2.21),(2.22),(2.20) to (2.19), we get

$$
\begin{aligned}
|E_K(fp)| &\le c\,det(B_K)\sum_{i=0}^{1}|\hat{f}|_{i,q,\hat{K}}|\hat{p}|_{1-i,\hat{K}} \\
&\le ch_k|det(B_K)|^{\frac{1}{2}-\frac{1}{q}}\|f\|_{1,q,K}\|p\|_{1,K}.
\end{aligned}
$$

Proof of Theorem 2.1.

Applying Lemma 2.5,2.6,2.8,2.9 to Lemma 2.4 directly yields Theorem 2.1.

Now we give some quadrature schemes satisfying Assumption 1 ,2 , 3 . For simplicity , we choose the reference rectangle $\hat{K}=[-1,1]^2$ with the vertices $A_1(1,1), A_2(1,-1), A_3(-1,-1), A_4(-1,1)$.

Scheme 1.

$$
\int_{\hat{K}}\hat{\phi}(\hat{x})d\hat{x} \approx \sum_{l=1}^{4}\hat{\phi}(A_l).
$$

It is a widely used scheme in numerical integration, which satisfies Assumption 1, 2, 3. Let $A_{12}(1,0), A_{23}(0,-1), A_{34}(-1,0), A_{41}(0,1)$ be the midpoints of four edges of $\hat{K}$, A_1A_2 , A_2A_3, A_3A_4 ,A_4A_1 respectively. Using one of the four midpoints together with the two vertices of its opposite edge, we derive a new scheme as follows:

Scheme 2.

$$
\int_{\hat{K}}\hat{\phi}(\hat{x})d\hat{x} \approx 2\hat{\phi}(A_{12})+\hat{\phi}(A_3)+\hat{\phi}(A_4).
$$

It satisfies also Assumption 1,2,3. Similarly, we can derive another three schemes using the midpoints of A_2A_3, A_3A_4, A_4A_1 respectively.

3.Multigrid Method for Wilson Nonconforming Element

In this section we describe the V-cycle multigrid method for Wilson nonconforming element. We construct two multigrid algorithms, which have the same convergence property as for conforming elements [1],[2].

Let $\mathcal{J}_{h_0},\cdots,\mathcal{J}_{h_J}$ be a sequence of rectangular partitions of Ω, satisfying the regularity assumption[2]. Suppose $\mathcal{J}_{h_k}$ is obtained by dividing each rectangle $K\in\mathcal{J}_{h_{k-1}}$ into four equal rectangles, $k=1,\cdots,J$. The corresponding Wilson element space W_{h_k} on $\mathcal{J}_{h_k}$ is denoted by W_k, then we get a sequence of nonnested finite-dimensional vector spaces $W_1, W_2,\cdots,W_J$. Let $\overline{A}_k(.,.), A_k(.,.)$ denote respectively $\overline{A}_{h_k}(.,.), A_{h_k}(.,.)$ defined in Section 2.

Given $f\in M_k$, find $v\in M_k$ satisfying

$$
\overline{A}_k(v,\phi)=(f,\phi) \quad \text{for all} \quad \phi\in W_k \tag{3.1a}
$$

We now define the V-cycle multigrid algorithm for (3.1a) as follows. First, we need an intergrid transfer operator $I_k : M_{k-1}\rightarrow M_k, k=0,1,\cdots,J$, which will be given later. We need also some auxiliary operators. For $k=0,1,\cdots,J$, we define the operator $\overline{A}_k : W_k\rightarrow W_k$ by

$$
(\overline{A}_k w,\phi)=\overline{A}_k(w,\phi), \quad \text{for all} \quad \phi\in W_k. \tag{3.1b}
$$

The operator $\overline{A}_k$ is clearly symmetric and positive definite. Then we define the operator $\overline{P}_{k-1}$: $W_k\rightarrow W_{k-1}$ and $\overline{Q}_{k-1} : W_k\rightarrow W_{k-1}$ by

$$
\overline{A}_{k-1}(\overline{P}_{k-1}w,\phi)=\overline{A}_k(w,I_k\phi) \quad \text{for all} \quad w\in M_k, \phi\in W_{k-1}. \tag{3.1c}
$$

and

$$(\overline{Q}_{k-1}w,\phi) = (w, I_k\phi) \quad \text{for all} \quad w \in M_k, \phi \in W_{k-1}. \tag{3.1d}$$

Moreover, We need a linear smoothing operator $R_k : W_k \longrightarrow W_k$ for $k = 1, \cdots, J$, and in addition we define

$$R_k^{(l)} = \begin{cases} R_k, & \text{if l is odd,} \\ R_k^t & \text{if l is even,} \end{cases}$$

where R_k^t is the adjoint of R_k with respect to the inner product $(.,.)$. The operator R_k satisfies certain conditions which will be stated later.

We can now define the multigrid operator $\overline{B}_k : W_k \longrightarrow W_K$ by induction.

The V-cycle Multigrid Algorithm.

Set $\overline{B}_0 = \overline{A}_0^{-1}$. Assume that $\overline{B}_{k-1}$ has been defined. We define $\overline{B}_k g$ for $g \in W_k$ as follows:

(1) Set $x^0 = 0$.

(2) Pre-smoothing. Define x^l for $l = 1, \cdots, m(k)$ by

$$x^l = x^{l-1} + R_k^{(l+m(k))}(g - \overline{A}_k x^{l-1}). \tag{3.2}$$

(3) Correction. Define $y^{m(k)} = x^{m(k)} + I_k q$, where q is defined by

$$q = \overline{B}_{k-1}\overline{Q}_{k-1}(g - \overline{A}_k x^{m(k)}). \tag{3.3}$$

(4) Post-smoothing. Define y^l for $l = m(k)+1, \cdots, 2m(k)$ by

$$y^l = y^{l-1} + R_k^{(l+m(k))}(g - \overline{A}_k y^{l-1}).$$

(5) Set $\overline{B}_k g = y^{2m(k)}$.

Here $m(k)$ is a positive integer which may vary from level to level and determines the number of smoothing iterations on that level. If $m(k)$ is a constant for all levels, the algorithm is called simply the V-cycle. Otherwise, it is the variable V-cycle.

It is straightforward to check that

$$I - \overline{B}_k\overline{A}_k = (\overline{K}_k^{m(k)})^*[(I - I_k\overline{P}_{k-1}) + I_k(I - \overline{B}_{k-1}\overline{A}_{k-1})\overline{P}_{k-1}]\overline{K}_k^{m(k)}, \tag{3.4}$$

where

$$\overline{K}_k^{(m)} = \begin{cases} (K_k^* K_k)^{\frac{m}{2}}, & \text{if m is even,} \\ (K_k^* K_k)^{\frac{m-1}{2}} K_k^*, & \text{if m is odd,} \end{cases}$$

$(\overline{K}_k^{m(k)})^*$ is the adjoint of $\overline{K}_k^{m(k)}$ with respect to $(.,.)$, $K_k^* = I - R_k^t\overline{A}_k$. For the convergence analysis, we need some assumptions. The first one is refered to the "regularity and approximation" assumption as follows [3]:

$$|\overline{A}_k((I - I_k\overline{P}_{k-1})u,u)| \le C_\alpha^2(\frac{\|\overline{A}_k u\|_k^2}{\lambda_k})^\alpha \overline{A}_k(u,u)^{1-\alpha} \quad \text{for all} \quad u \in M_k, \tag{A.1}$$

where λ_k is the largest eigenvalue of $\overline{A}_k$. C_α is independent of k for $k = 1, \cdots, J$, and $\alpha \in (0,1)$.

The second assumption is about the smoothing operator R_k [2]:

Let $K_k = I - R_k\overline{A}_k, K_{k,w} = I - w\lambda_k^{-1}\overline{A}_k$, there exists $w \in (0,1)$ such that

$$A(K_k v, K_k v) \le A(K_{k,w}v, v),$$
$$A(K_k^* v, K_k^* v) \le A(K_{k,w}v, v). \tag{A.2}$$

Under the condition (A.2), the operator $\overline{B}_k$ corresponding to the variable V-cycle or the V-cycle multigrid algorithm is positive definite and hence can be used as a preconditioner in an iterative method for solving (3.1a).The convergence rate of the iterative method depends on the bounds of the largest and smallest eigenvalues of the operator $\overline{B}_k\overline{A}_k$. Equivalently, we will provide two positive constants η_0 and η_1, which may depend on k and satisfy

$$\eta_0\overline{A}_k(u,u) \le \overline{A}_k(\overline{B}_k\overline{A}_k u, u) \le \eta_1\overline{A}_k(u,u) \quad \text{for all} \quad u \in M_k. \tag{3.5}$$

Note that if (3.5) holds, then the PCG method converges with an asymptotic rate of

$$\frac{1-\sqrt{\frac{\eta_0}{\eta_1}}}{1+\sqrt{\frac{\eta_0}{\eta_1}}}$$

per iterative step.

The next theorem provides estimates for η_0 and η_1 for the variable V-cycle algorithm.

Theorem 3.1[3]. Assume that (A.1), (A.2) hold and that $m(k)$ satisfies

$$\beta_0 m(k) \le m(k-1) \le \beta_1 m(k), \tag{3.6}$$

where β_0, β_1 are positive constants, greater than one and independent of k. Then (3.5) holds with η_0, η_1 satisfying

$$\eta_0 \ge \frac{m(k)^\alpha}{M+m(k)^\alpha} \quad \text{and} \quad \eta_1 \le \frac{M+m(k)^\alpha}{m(k)^\alpha}, \tag{3.7}$$

where M is a constant independent of k and $m(k)$.

Corollary. The condition number of the matrix $\overline{B}_k\overline{A}_k$ is bounded.

It means the matrix $\overline{B}_k$ is a good preconditioner for the matrix $\overline{A}_k$.

Now we construct the intergrid transfer operator $I_k : W_{k-1} \to W_k$ satisfying the regularity and approximation assumption (A.1).

Let M be a rectangle in $\mathcal{J}_{k-1}$, as shown in Figure 3.1. a_1, a_2, a_3, a_4 are its vertices , $c_0(x_0, y_0)$ is the center, b_1, b_2, b_3, b_4 are the midpoints of four edges. Joining b_1, b_3 and b_2, b_4, we get four equal rectangles M_1, M_2, M_3, M_4 in $\mathcal{J}_k$ with c_1, c_2, c_3, c_4 being their center. The length of a_1a_2, a_1a_4 is denoted by $2h_1, 2h_2$, $|M|$ is the area of M.

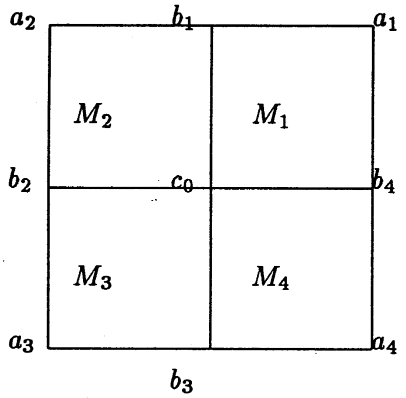

Fig.3.1.

For every $v \in W_{k-1}$, we define $I_k v$ on M_1 as follows:

$$
\begin{aligned}
& I_k v(c_0) = v(c_0), \\
& I_k v(a_1) = v(a_1), \\
& I_k v(b_1) = \frac{1}{2}(v(a_1) + v(a_2)), \\
& I_k v(b_4) = \frac{1}{2}(v(a_1) + v(a_4)), \\
& \frac{1}{|M_1|}\int_{M_1} \partial_{xx} I_k v dxdy = \frac{1}{|M|}\int_M \partial_{xx} v dxdy, \\
& \frac{1}{|M_1|}\int_{M_1} \partial_{yy} I_k v dxdy = \frac{1}{|M|}\int_M \partial_{yy} v dxdy.
\end{aligned} \tag{3.8}
$$

Similarly, we can define $I_k v$ on M_2, M_3, M_4, respectively.

The following fact is obvious: given $u \in W_k$,there exits $f_u \in W_k$, such that

$$\overline{A}_k(u,v) = (f_u, v) \quad \text{for all} \quad v \in W_k. \tag{3.9}$$

Let $u^* \in H_0^1(\Omega), w \in BL_k$ be the solution of the following equations, respectively:

$$\overline{A}(u^*, v) = (f_u, v) \quad \text{for all} \quad v \in H_0^1(\Omega), \tag{3.10}$$

$$\overline{A}(w, v) = (f_u, v) \quad \text{for all} \quad v \in BL_k, \tag{3.11}$$

where $BL_k \subset W_k$ is the bilinear element space.

Lemma 3.1[7].

$$\overline{A}_k(u,v) \le C_1 ||u||_{1,k} ||v||_{1,k},$$

$$C_2 ||u||_{1,k}^2 \le \overline{A}_k(u,u),$$

$$\lambda_k = O(h_k^{-2}),$$

where $u, v \in W_k, \lambda_k$ is the largest eigenvalue of $\overline{A}_k$. C_1, C_2 are constants independent of K,. $||.||_{1,k}$ is the discrete H^1 norm on W_k.

Lemma 3.2.

$$|I_k v|_{1,k} \le c|v|_{1,k-1} \quad \text{for all} \quad v \in W_{k-1}.$$

Proof. Let $v = v^I + Z$, v^I is the bilinear interpolant of v in Q_k, Z is the nonconforming part of v, M be a rectangle in $\mathcal{J}_{k-1}$.

By a simple computation, we get

$$\int_M \nabla v^I \nabla Z = 0,$$

therefore,

$$|v|_{1,k-1}^2 = |v^I|_1^2 + |Z|_{1,k-1}^2. \tag{3.12}$$

¿From the definition of I_k, we have

$$I_k Z(b_1) = 0.$$

Let ϕ_x, ϕ_y denote the terms appearing in the right side of the last two equations of (3.8). We have

$$Z(b_1) = -\frac{h_1^2}{2}\phi_x,$$

thus

$$(Z - I_kZ)(b_1) = -\frac{h_1^2}{2}\phi_x.$$

Similarly,

$$(Z - I_kZ)(b_2) = -\frac{h_2^2}{2}\phi_y.$$

Moreover,

$$(Z - I_kZ)(a_2) = (Z - I_kZ)(c_0) = 0,$$

and its two second derivatives on M_2 are also zero, thus on M_2 we have

$$\begin{aligned} Z - I_kZ = &-\frac{h_1^2}{2}\phi_x\frac{(x-x_0)(y-y_0-h_2)}{h_1h_2} \\ &-\frac{h_2^2}{2}\phi_y\frac{(x-x_0-h_1)(y-y_0-2h_2)}{h_1h_2}. \end{aligned}$$

A further computation gives

$$\begin{aligned} |Z - I_kZ|_{1,M_2}^2 &\le 2(\phi_x^2+\phi_y^2)(\frac{2hk^3}{3}+\frac{2h^3k}{3}) \\ &= |Z|_{1,M}^2. \end{aligned}$$

Similarly, we can prove

$$|Z - I_kZ|_{1,M_i}^2 \le |Z|_{1,M}^2, \quad i = 1,3,4.$$

Therefore,

$$|Z - I_kZ|_{1,K} \le c|Z|_{1,k-1}. \tag{3.13}$$

Meanwhile, the continuity of v^I implies

$$|I_kv^I|_1 = |v^I|_1. \tag{3.14}$$

Using (3.12),(3.13) and (3.14), we have

$$\begin{aligned} |I_kv|_{1,k}^2 &\le 2|I_kv^I|_{1,k}^2 + |I_kZ|_{1,k}^2 \\ &\le 2|v^I|_{1,k}^2 + 2|Z|_{1,k}^2 + 2|Z - I_kZ|_{1,k}^2 \\ &\le c(|v^I|_{1,k-1}^2 + |Z|_{1,k-1}^2) \\ &= c|v|_{1,k-1}^2, \end{aligned}$$

which completes the proof.

Lemma 3.3.

$$\|u - I_k\overline{P}_{k-1}u\|_{1,k} \le ch_k\|\overline{A}_ku\|_0 \quad \text{for all} \quad u \in W_k.$$

Proof. Let w_{k-1}, u_{k-1} denote respectively the solution of the following equations:

$$\overline{A}(w_{k-1}, v) = (f_u, v) \quad \text{for all} \quad v \in BL_{k-1}, \tag{3.15}$$

$$\overline{A}_{k-1}(u_{k-1}, v) = (f_u, v) \quad \text{for all} \quad v \in W_{k-1}, \tag{3.16}$$

where $f_u = \overline{A}_ku$ and u^* are defined in (3.9) and (3.10). The elliptic regularity follows

$$\|u^*\|_{2,\Omega} \le c\|f_u\|_0. \tag{3.17}$$

Therefore,

$$\begin{aligned}
&\|u - I_k\overline{P}_{k-1}u\|_{1,k} \\
&\le \|u - w_{k-1}\|_{1,k} + \|I_k(w_{k-1} - \overline{P}_{k-1}u)\|_{1,k} \\
&\le \|u^* - u\|_{1,k} + \|u^* - w_{k-1}\|_{1,k-1} + c\|w_{k-1} - \overline{P}_{k-1}u\|_{1,k-1} \\
&\le \|u^* - u\|_{1,k} + \|u^* - w_{k-1}\|_{1,k-1} \\
&+ c(\|u_{k-1} - w_{k-1}\|_{1,k-1} + \|u_{k-1} - \overline{P}_{k-1}u\|_{1,k-1}).
\end{aligned} \tag{3.18}$$

The finite element error estimates and (3.17) give

$$\|u^* - u\|_{1,k} \le ch\|u^*\|_2 \le ch\|f_u\|_0 = ch\|\overline{A}_k u\|_0, \tag{3.19}$$

$$\|u^* - w_{k-1}\|_{1,k-1} \le ch\|u^*\|_2 \le ch\|\overline{A}_k u\|_0, \tag{3.20}$$

$$\begin{aligned}
\|u_{k-1} - w_{k-1}\|_{1,k-1} &\le \|u^* - u_{k-1}\|_{1,k-1} + \|u^* - w_{k-1}\|_{1,k-1} \\
&\le ch\|\overline{A}_k u\|_0.
\end{aligned} \tag{3.21}$$

From the definition of $\overline{P}_{k-1}$, we have

$$\begin{aligned}
\overline{A}_{k-1}(\overline{P}_{k-1}u, v) &= A_k(u, I_k v) \\
&= (f_u, I_k v) \quad \text{for all} \quad v \in W_{k-1},
\end{aligned} \tag{3.22}$$

which together with (3.16) gives

$$\begin{aligned}
\|u_{k-1} - \overline{P}_{k-1}u\|_{1,k-1} &= sup_{v\in W_{k-1}} \frac{|\overline{A}_{k-1}(\overline{P}_{k-1}u - u_{k-1}, v)|}{\|v\|_{1,k-1}} \\
&= sup_{v\in W_{k-1}} \frac{|(f_u, v - I_k v)|}{\|v\|_{1,k-1}} \\
&\le sup_{v\in W_{k-1}} \frac{\|v - I_k v\|_0}{\|v\|_{1,k-1}} \|f_u\|_0,
\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}
\|v - I_k v\|_0 &\le \|v - (I_k v)^I\|_0 + \|I_k v - (I_k v)^I\|_0 \\
&\le ch\|v\|_{1,k-1} + ch\|I_k v\|_{1,k} \\
&\le ch\|v\|_{1,k-1}.
\end{aligned} \tag{3.24}$$

Therefore, we get

$$\|u_{k-1} - \overline{P}_{k-1}u\|_{1,k-1} \le ch\|A_k u\|_0. \tag{3.25}$$

Combining (3.19),(3.20),(3.21),(3.25) with (3.18), we complete the proof of this lemma.

Lemma 3.1 and Lemma 3.3 immediately imply

Lemma 3.4. Let I_k be defined as before, then (A1) holds with $\alpha = \frac{1}{2}$.

From Lemma 3.4 and Theorem 3.1, we obtain the main result of this section as follows:

Theorem 3.2. The multigrid algorithm is defined as before with $R_k, m(k)$ suitably chosen and the integrid transfer operator I_k is defined as (3.8). Then (3.5) holds with η_0, η_1 satisfying (3.7), where $\alpha = \frac{1}{2}$.

Now, we construct another integrid operator $\tilde{I}_k$, which has a better convergence property than the previous one, I_k.

Let $\tilde{I}_k$ be such that at the center c_0, it takes the value

$$\tilde{I}_k(c_0) = \frac{1}{4}(v(a_1) + v(a_2) + v(a_3) + v(a_4)),$$

and the other five definitions are the same as those of I_k in (3.8).

For this new intergrid transfer operator $\tilde{I}_k$, we have

$$Z - \tilde{I}_k Z = (Z - I_k Z) + (-\frac{h_1^2}{2}\phi_x - \frac{h_2^2}{2}\phi_y)\frac{(x-x_0)(y-y_0-2h_2)}{h_1 h_2},$$

thus

$$|Z - \tilde{I}_k Z|_{1,k} \le c|Z|_{1,k-1}.$$

Comparing with (3.13), it is seen that Lemma 3.2 is valid for $\tilde{I}_k$. Therefore, Lemma 3.3 and 3.4 also hold for this new transfer operator.

We have a similar Theorem for $\tilde{I}_k$.

Theorem 3.3. The multigrid algorithm is defined as before with $R_k, m(k)$ suitably chosen and the intergrid transfer operator $\tilde{I}_k$ is defined above. Then (3.5) holds with η_0, η_1 satisfying (3.7), where $\alpha = \frac{1}{2}$.

It will be seen later that $\tilde{I}_k$ has a better convergence property than I_k. In fact, it comes from an important property which $\tilde{I}_k$ has but I_k hasn't as follows:

$$\overline{A}_k(\tilde{I}_k u, \tilde{I}_k u) \le \overline{A}_{k-1}(u,u) \quad \text{for all} \quad u \in W_{k-1}. \tag{A.3}$$

Theorem 3.4[3]. Assume (A.1),(A.2) and (A.3) hold. $\overline{B}_k$ is defined as before and $m(k)$ satisfies (3.6). Then

$$|\overline{A}_k((I - \overline{B}_k\overline{A}_k)u, u)| \le \delta_k \overline{A}_k(u,u) \quad \text{for all} \quad u \in W_k \tag{3.26}$$

holds with

$$\delta_k = \frac{M}{M + m(k)^\alpha}. \tag{3.27}$$

Remark 3.1. If $\tilde{I}_k$ satisfies (A.3), then the new preconditioner $\overline{B}_K$ will satisfy (3.26), which is obviously stronger than (3.5). Indeed, this $\overline{B}_k$ can be directly used as an iterative operator, besides as a preconditioner, that will speed up the convergence procedure.

For simplicity, we assume that $a_{ij} = \delta_{ij}, a = 0$. It means that the original equation is Poisson equation $-\Delta u = f$.

Lemma 3.5. (A.3) holds for $\tilde{I}_k$.

Proof. See Figure 3.1. Let $\phi_i = v(a_i), 1 \le i \le 4, \phi_0 = v(c_0), v = v^I + Z$, then

$$\tilde{I}_k v = v^I + \tilde{I}_k Z,$$

and

$$\begin{aligned}\overline{A}_{k-1}(v,v) &= \sum_{M \in \mathcal{J}_{k-1}} \int_M (\partial_x v)^2 + (\partial_y v)^2 dxdy \\ &= \sum_{M \in \mathcal{J}_{k-1}} (\int_M (\partial_x v^I)^2 + (\partial_y v^I)^2 dxdy + \int_M (\partial_x Z)^2 + (\partial_y Z)^2 dxdy).\end{aligned}$$

The last equation is from (3.12), the term after $\sum$ is denoted by $A_{k-1}^M v$.

By a careful computation, we have

$$\begin{aligned}A_{k-1}^M v = &\frac{h_2}{3h_1}[(\phi_1-\phi_2)^2+(\phi_3-\phi_4)^2-(\phi_1-\phi_2)(\phi_3-\phi_4)]\\ &+\frac{h_1}{3h_2}[(\phi_1-\phi_4)^2+(\phi_3-\phi_2)^2-(\phi_1-\phi_4)(\phi_3-\phi_2)]\\ &+\frac{4}{3}h_1h_2(h_1^2\phi_x^2+h_2^2\phi_y^2).\end{aligned}$$

Meanwhile,

$$\overline{A}_k(\tilde{I}_k v,\tilde{I}_k v)=\sum_{M\in\mathcal{J}_{k-1}}\sum_{i=1}^{4}\int_{M_i}(\partial_x\tilde{I}_k v)^2+(\partial_y\tilde{I}_k)^2dxdy.$$

Let $A_k^{M_i}$ denote the integral on the right side of the last equation. we can calculate $A_k^{M_1}$, which gives

$$\begin{aligned}A_k^{M_1}v = &\frac{1}{3}[\frac{\frac{h_1}{h_2}(\phi_2-\phi_3)^2+\frac{h_1}{h_2}(\phi_4-\phi_3)^2+7\frac{h_2}{h_1}(\phi_4-\phi_1)^2+7\frac{h_2}{h_1}(\phi_2-\phi_1)^2}{16}\\ &+\frac{1}{4}\frac{h_1}{h_2}(\phi_1-\phi_4)(\phi_2-\phi_3)+\frac{1}{4}\frac{h_2}{h_1}(\phi_1-\phi_2)(\phi_4-\phi_3)]\\ &\frac{1}{12}h_1h_2(h_1^2\phi_x^2+h_2^2\phi_y^2).\end{aligned}$$

Other $A_k^{M_i}$ can be calculated similarly. Therefore,

$$\begin{aligned}\sum_{i=1}^{4}A_k^{M_i}v = &\frac{h_2}{3h_1}[(\phi_1-\phi_2)^2+(\phi_3-\phi_4)^2-(\phi_1-\phi_2)(\phi_3-\phi_4)]\\ &+\frac{h_1}{3h_2}[(\phi_1-\phi_4)^2+(\phi_3-\phi_2)^2-(\phi_1-\phi_4)(\phi_3-\phi_2)]\\ &+\frac{1}{3}h_1h_2(h_1^2\phi_x^2+h_2^2\phi_y^2)\\ &\le A_{k-1}^M v,\end{aligned}$$

thus we have

$$\overline{A}_k(\tilde{I}_k v,\tilde{I}_k v)\le\overline{A}_{k-1}(v,v)\quad\text{for all}\quad v\in W_{k-1}.$$

Lemma 3.5 implies

Theorem 3.5. The multigrid algorithm is defined as before with $m(k), R_k$ suitably chosen and the intergrid transfer operator $\tilde{I}_k$ is defined as before. Then (3.26) holds with δ_k satisfying (3.27).

4.Multigrid With Numerical Integration Method for Wilson Nonconforming Element

Combining the results of Section 2 and Section 3, we can start our discussion on multigrid method for Wilson nonconforming element, when a proper quadrature scheme is used for the approximation. We will show that the preconditioner constructed by using a suitable quadrature scheme has the same effect as that in Section 3.

It was proved in[8] that when a quadrature scheme satisfying

Assumption 4[7]:

$$E_K[\phi]\equiv 0\quad\forall\quad\phi\in P_2(K)$$

is used in the multigrid method for a conforming element, we can get a good preconditioner as without numerical integration. However, Assumption 4 is stronger than Assumption 1,2,. For example, the five integration schemes proposed in Section 2 for Wilson element don't satisfy Assumption 4. However, we have proved in [13] that using a scheme satisfying the Assumption 1,2,3, but not the Assumption 4, we can still get the same good preconditioner for conforming elements as in [8]. In this section, we will prove that these five schemes defined above can also be used for Wilson nonconforming element.

We define the multigrid algorithm with numerical integration just the same as the multigrid algorithm in section 3, with $\overline{B}_k, \overline{A}_k$ replaced by B_k, A_k, respectively. A_k, P_k, Q_k are defined in (3.1b,c,d) with $\overline{A}_k$ replaced by A_k. B_k is the preconditioner for A_k,λ_k is the largest eigenvalue of A_k.

Using the knowledge in [7], we can prove

Lemma 4.1. Let Assumption 1,2,3 hold. Then there exist positive constants c and c' independent of k, such that

$$c^{-1}\overline{A}_k(u,u) \le A_k(u,u) \le c\overline{A}_k(u,u) \quad \text{for all} \quad u \in W_k, \tag{4.1}$$

$$(c')^{-1}||u||_{1,k} \le c^{-1}||u||_{\overline{A}_k} \le c||u||_{A_k} \le c||u||_{\overline{A}_k} \le c'||u||_{1,k} \quad \text{for all} \quad u \in W_k, \tag{4.2}$$

$$(c')^{-1}h_k^{-2} \le c^{-1}\overline{\lambda}_k \le c\overline{\lambda}_k \le c'h_k^{-2}, \tag{4.3}$$

$$c^{-1}||A_k u||_k \le ||u||_{A_k} \le c||A_k u||_k \quad \text{for all} \quad u \in W_k. \tag{4.4}$$

Lemma 4.2. Suppose Assumption 1,2,3 hold and $a, a_{ij} \in W^{1,\infty}(\Omega), i,j = 1,2$. Let u^* denote the solution of (2.2) with $f = A_k u$, then

$$||u - u^*||_{1,k} \le ch_k \sum_{i,j=1}^{2} (||a_{ij}||_{1,\infty} + ||a||_{1,\infty})||u^*||_2. \tag{4.5}$$

Proof. It is clear that

$$\overline{A}(u^*, v) = (A_k u, v) \quad \text{for all} \quad v \in H^1(\Omega), \tag{4.6a}$$

$$A_k(u, v) = (A_k u, v) \quad \text{for all} \quad v \in W_k, \tag{4.6b}$$

where A_k is the numerical integration approximation to $\overline{A}$. Applying the first Strang Lemma[7], we have

$$\begin{aligned} ||u^* - u||_{1,k} \le c[inf_{v_h \in W_k}(||u^* - v_h||_{1,k} + sup_{w_h \in W_k} \frac{|\overline{A}_k(v_h, w_h) - A_k(v_h, w_h)|}{||w_h||_{1,k}}) \\ + sup_{w_h \in W_k} \frac{|\overline{A}_k(u^*, w_h) - (f, w_h)|}{||w_h||_{1,k}}]. \end{aligned} \tag{4.7}$$

Let $\overline{u}$ be the interpolation of u^* at the nodes. Applying the standard interpolation error estimates to the first term on the right side of (4.7), we have

$$||u^* - \overline{u}||_{1,k} \le ch_k||u^*||_2, \tag{4.8}$$

$$||u^* - \overline{u}||_{H^2(K)} \le c||u^*||_{H^2(\tau_k^i)}. \tag{4.9}$$

Then, applying Lemma 2.7 and (4.9) to the second term on the right side of (4.7), we get

$$\begin{aligned} |\overline{A}_k(\overline{u}, w_h) - A_k(\overline{u}, w_h)| &\le ch_k \sum_{i,j=1}^{2} (||a_{ij}||_{1,\infty} + ||a||_{1,\infty})(\sum_{K \in T_k} ||\overline{u}||^2_{H^2(\tau_k^i)})^{\frac{1}{2}}||w_h||_{1,k} \\ &\le ch_k \sum_{i,j=1}^{2} (||a_{ij}||_{1,\infty} + ||a||_{1,infty})||u^*||_2||w_h||_{1,k}. \end{aligned} \tag{4.10}$$

On the other hand, the consistency error estimate in [7] gives

$$sup_{w_h \in W_k} \frac{|\overline{A}_k(u^*, w_h) - (f, w_h)|}{||w_h||_{1,k}} \leq ch||u^*||_2.$$

Combination of (4.7),(4.8),(4.10) and the last inequality completes the proof.

Lemma 4.3. Suppose $a, a_{ij} \in W^{1,\infty}(\Omega)$, and Assumption 1,2,3 hold. Then for all $u, v \in W_k$,

$$|\overline{A}_k(u,v) - A_k(u,v)| \leq ch_k||v||_{1,k}||\overline{A}_k u||_0, \tag{4.11a}$$

$$|\overline{A}_k(u,v) - A_k(u,v)| \leq ch_k||v||_{1,k}||A_k u||_0. \tag{4.11b}$$

Proof. From Lemma 2.7, we have

$$|\overline{A}_k(u,v) - A_k(u,v)| \leq ch_k||v||_{1,k}(\sum_{\tau_k^i \in T_k} ||u||^2_{H^2(K)})^{\frac{1}{2}}. \tag{4.12}$$

Let u^* denote the solution of (3.1a,b) with $f = \overline{A_k}u$. Using Wilson element error estimate, we have

$$||u^* - u||_{1,k} \leq ch||u^*||_2. \tag{4.13}$$

The full elliptic regularity yields

$$||u^*||_2 \leq c||\overline{A}_k u||_0. \tag{4.14}$$

Let $\overline{u}$ be the interpolation of u^* at the nodes. It follows that

$$||u^* - \overline{u}||_{H^1(K)} \leq ch_k||u^*||_{H^2(K)}, \tag{4.15}$$

$$||\overline{u}||_{H^2(K)} \leq c||u^*||_{H^2(K)}. \tag{4.16}$$

Hence

$$\begin{aligned} ||u||^2_{H^2(K)} &\leq 2||u - \overline{u}||^2_{H^2(K)} + 2||\overline{u}||^2_{H^2(K)} \\ &\leq c(h_k^{-2}||u - \overline{u}||^2_{H^1(K)} + ||u^*||^2_{H^2(K)}) \\ &\leq ch_k^{-2}||u^* - u||^2_{H^1(K)} + ||u^*||^2_{H^2(K)}. \end{aligned} \tag{4.17}$$

Taking the sum over all elements and using (4.13),(4.14), it follows (4.11a).

(4.11b) can be proved similarly by noting that (4.13) can be replaced by Lemma 4.2.

By application of the above three Lemmas, we can prove the following lemma easily.

Lemma 4.4. Suppose $a, a_{ij} \in W^{1,\infty}(\Omega)$, and Assumption 1,2,3 hold.Then for all $u \in W_k$,

$$C^{-1}||A_k u||_0 \leq ||\overline{A}_k u||_0 \leq c||A_k u||_0, \tag{4.18a}$$

$$C^{-1}\lambda_k^{-1}||A_k u||_0 \leq \overline{\lambda}_k^{-1}||\overline{A}_k u||_0 \leq c\lambda_k^{-1}||A_k u||_0. \tag{4.18b}$$

Lemma 4.5. Suppose $a, a_{ij} \in W^{1,\infty}(\Omega)$, and Assumption 1,2,3 hold. Then for all $u \in W_k$,

$$\begin{aligned} ||(\overline{P}_{k-1} - P_{k-1})u||_{1,k-1} &\leq ch_k||A_k u||_0 \\ &\leq ch_k||\overline{A}_k u||_0, \end{aligned} \tag{4.19}$$

$$||\overline{P}_{k-1}u||_{1,k-1} \leq c||u||_{1,k}, \tag{4.20a}$$

$$||P_{k-1}u||_{1,k-1} \leq c||u||_{1,k}. \tag{4.20b}$$

Proof. From the definition of $\overline{P}_{k-1}, P_{k-1}$, for all $u \in W_k, v \in W_{k-1}$, we have

$$A_{k-1}((\overline{P}_{k-1} - P_{k-1})u, v)$$
$$\leq |A_{k-1}(\overline{P}_{k-1}u, v) - \overline{A}_{k-1}(\overline{P}_{k-1}u, v)| + |A_k(u, I_k v) - \overline{A}_k(u, I_k v)|.$$

Let I_1, I_2 denote the two terms in the right side of the last inequality. Using Lemma 4.3 and Lemma 3.2, we have

$$I_2 \leq ch_k ||A_k u||_0 ||I_k v||_{1,k} \leq ch_k ||A_k u||_0 ||v||_{1,k-1},$$

$$\begin{aligned} I_1 &\leq ch_{k-1} ||\overline{A}_{k-1}\overline{P}_{k-1}u||_0 ||v||_{1,k-1} \\ &\leq ch_k ||\overline{Q}_{k-1}\overline{A}_k u||_0 ||v||_{1,k-1} \\ &\leq ch_k ||\overline{A}_k u||_0 ||v||_{1,k-1}. \end{aligned}$$

Therefore,

$$||\overline{P}_{k-1} - P_{k-1}||_{1,k-1} \leq ch_k ||A_k u||_0 \leq ch_k ||\overline{A}_k u||_0$$

and

$$\begin{aligned} ||\overline{P}_{k-1}||_{1,k-1} &= sup_{v \in W_{k-1}} \frac{|\overline{A}_{k-1}(\overline{P}_{k-1}u, v)|}{||v||_{1,k-1}} \\ &= sup_{v \in W_{k-1}} \frac{|\overline{A}_k(u, I - kv)|}{||v||_{1,k-1}} \\ &\leq sup_{v \in W_{k-1}} \frac{c||u||_{1,k} ||I_k v||_{1,k}}{||v||_{1,k-1}} \\ &\leq c||u||_{1,k}. \end{aligned}$$

Similarly, (4.20b) can be proved.

Now we turn to the proof of the main condition (A.1) in Section 3, when a quadrature scheme satisfying Assumption 1,2,3 is used. From now on, when we mention the condition (A.1) or (3.5), we always suppose $\overline{A}_k, \overline{P}_{k-1}$ are replaced by A_k, P_{k-1}, since only the numerical integration methods are considered. If (A.1) holds and the smoother R_k is well chosen, then we can obtain the same good preconditioners for the variable V-cycle algorithm as those in section 3.

Theorem 4.1. Suppose $a, a_{ij} \in W^{1,\infty}(\Omega)$,the multigrid algorithm is defined as before, and Assumption 1,2,3 hold. Then (A.1) holds with $\alpha = \frac{1}{2}$.
Proof. Theorem 3.2, Lemma 4.1 and Lemma 4.4 give

$$\begin{aligned} |\overline{A}((I - I_k\overline{P}_{k-1})u, u)| &\leq C_\alpha(\overline{\lambda}_k^{-1} ||\overline{A}_k u||_0^2)^{\frac{1}{2}} \overline{A}_k(u, u)^{\frac{1}{2}} \\ &\leq C_\alpha(\lambda_k^{-1} ||A_k u||_0^2)^{\frac{1}{2}} A_k(u, u)^{\frac{1}{2}}. \end{aligned} \tag{4.21}$$

Using Lemma 4.3, Lemma 3.2 and Lemma 4.5, we have

$$|A_k((I - I_k P_{k-1})u, u) - \overline{A}((I - I_k P_{k-1})u, u)| \leq ch_k ||u||_{1,k} ||A_k u||_0, \tag{4.22}$$

which follows

$$\begin{aligned} |A_k((I - P_{k-1})u, u)| \leq\; & ch_k ||u||_{1,k} ||A_k u||_{L_2} \\ & + |\overline{A}((I - I_k\overline{P}_{k-1})u, u)| + |\overline{A}(I_k(\overline{P}_{k-1} - P_{k-1})u, u)|. \end{aligned} \tag{4.23}$$

Applying (4.21) to the second term, and (4.11b), Lemma 3.2 and (4.19) to the last term on the right side of (4.23), we get

$$|A_k((I - I_k P_{k-1})u, u)| \le ch_k \|u\|_{1,k} \|A_k u\|_0 + c(\lambda_k^{-1} \|A_k u\|_0^2)^{\frac{1}{2}} A_k(u,u)^{\frac{1}{2}}.$$

Finally, using Lemma 4.1, we can see (A.1) holds with $\alpha = \frac{1}{2}$.
The following theorem is a consequence of Theorem 4.1.

Theorem 4.2. Suppose $a, a_{ij} \in W^{1,\infty}(\Omega)$, the multigrid algorithm is defined as before and the quadrature scheme satisfying Assumption 1,2,3 is used in approximations. Then (3.5) holds with η_0, η_1 satisfying (3.7).

Now we will examine whether the condition (A.3) holds for the intergrid transfer operator $\tilde{I}_k$ when the numerical quadrature schemes satisfying Assumption 1, 2, 3, for example, scheme 1 and scheme 2 in Section 2 are used in the approximation of Poisson equation. If it holds, the corresponding preconditioner B_k has the same good convergence property as $\overline{B}_k$ without using numerical integration.

Lemma 4.8. Assume the quadrature scheme 1 or 2 is used in approximations and $\tilde{I}_k$ is defined as before. Then

$$A_k(\tilde{I}_k v, \tilde{I}_k v) \le A_{k-1}(v,v) \quad \text{for all} \quad v \in W_{k-1}.$$

Proof. We prove the Lemma for the scheme 1. All notations are the same as Lemma 3.5. By a careful computation, we have

$$A_{k-1}(v,v) = \sum_{M \in \mathcal{J}_{k-1}} A_{k-1}^M v,$$

where

$$\begin{aligned} A_{k-1}^M v &= \sum_{i=1}^4 [\partial_x^2 v(a_i) + \partial_y^2 v(a_i)] h_1 h_2 \\ &= (4h_1^2 \phi_x^2 + 4h_2^2 \phi_y^2 + \frac{(\phi_1 - \phi_2)^2 + (\phi_4 - \phi_3)^2}{2h_1^2} \\ &+ \frac{(\phi_1 - \phi_4)^2 + (\phi_2 - \phi_3)^2}{2h_2^2}) h_1 h_2 \end{aligned}$$

and

$$A_k(\tilde{I}_k v, \tilde{I}_k v) = \sum_{M \in \mathcal{J}_{k-1}} \sum_{i=1}^4 A_k^{M_i} v,$$

where

$$\begin{aligned} A_k^{M_1} v = (h_1^2 \phi_x^2 + h_2^2 \phi_y^2 &+ \frac{\frac{5}{4}(\phi_1 - \phi_2)^2 + \frac{1}{4}(\phi_4 - \phi_3)^2 + \frac{1}{2}(\phi_1 - \phi_2)(\phi_4 - \phi_3)}{2h_1^2} \\ &+ \frac{\frac{5}{4}(\phi_1 - \phi_4)^2 + \frac{1}{4}(\phi_2 - \phi_3)^2 + \frac{1}{2}(\phi_1 - \phi_4)(\phi_2 - \phi_3)}{2h_2^2}) \frac{h_1 h_2}{4}. \end{aligned}$$

Similarly, other $A_k^{M_i} v$ can be calculated.

Therefore,

$$\sum_{i=1}^{4} A_k^{M_i} v = (4h_1^2\phi_x^2 + 4h_2^2\phi_y^2 + \frac{3(\phi_1-\phi_2)^2 + 3(\phi_4-\phi_3)^2 + (\phi_1-\phi_2)(\phi_4-\phi_3)}{2h_1^2}$$

$$\frac{3(\phi_1-\phi_4)^2 + 3(\phi_2-\phi_3)^2 + (\phi_1-\phi_4)(\phi_2-\phi_3)}{2h_2^2})\frac{h_1h_2}{4}$$

$$\leq (h_1^2\phi_x^2 + h_2^2\phi_y^2 + \frac{(\phi_1-\phi_2)^2 + (\phi_4-\phi_3)^2}{2h_1^2}$$

$$+ \frac{(\phi_1-\phi_4)^2 + (\phi_2-\phi_3)^2}{2h_2^2})h_1h_2$$

$$\leq A_{k-1}^M v,$$

thus we have

$$A_k(\tilde{I}_k v, \tilde{I}_k v) \leq A_{k-1}(v,v) \quad \text{for all} \quad u \in W_{k-1}.$$

Using the same idea, we can prove that the Lemma is valid also for the quadrature scheme 2.
Lemma 4.8 together with Theorem 3.4 imply

Theorem 4.3. Suppose $a_{ij} = \delta_{ij}, a = 0$, the multigrid algorithm and the transfer operator $\tilde{I}_k$ are defined as before, and the quadrature scheme 1 or 2 is used in approximations. Then

$$A_k((I - B_k A_k)u, u) \leq \delta_k A_k(u,u) \quad \text{for all} \quad u \in W_k$$

holds with

$$\delta_k = \frac{M}{M + m(k)^\alpha}.$$

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